

Migrating Carrollian particles on magnetized black hole horizons

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By considering a misaligned (asymptotically uniform) magnetic field in the background of a rotating black hole, we uncover a possibility for a highly nontrivial motion of Carrollian particles on the black hole horizon that is characterized by a time-dependent velocity field and reminds us (because of its latitudinal oscillations) of a “monarch butterfly migration.”

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I. INTRODUCTION

Carrollian particles are massless particles that live on a null surface equipped with a Carroll structure, a horizon of a black hole, for example. Contrary to Carroll fluids that are known to feature nontrivial dynamics, e.g., [1–4], it was believed that a single Carroll particle cannot move [1,5,6]. However, as recently shown this is no longer true in (2 + 1)-dimensional Carrollian settings in the presence of magnetic fields [7,8]. Namely, considering a *double central extension* of the Carroll group, the massless Carroll particle with anyonic spin can move under the influence of a *magnetic field* B , according to the following equation:

$$\frac{dx^A}{dv} = \frac{\mu\chi}{\kappa_{\text{mag}}} \epsilon^{AB} \partial_B B, \quad (1)$$

where v is the (preferred on the horizon) Carrollian time. Parametrizing the 2-dimensional particle’s trajectory $x^A = x^A(v)$, $A = 1, 2$, μ is the magnetic moment of the particle, χ its anyonic spin, κ_{mag} is a central extension parameter which allows for the particle to couple to electromagnetism, and ϵ^{AB} is the 2-dimensional Levi-Civita tensor of the 2-dimensional metric on the horizon (including its determinant).

For its remarkable similarity with the following (effectively 2-dimensional) 3D equation:

$$\frac{dx^i}{dv} = -e(E \times \Theta)^i, \quad (2)$$

of the spin Hall effect, e.g., [9,10] (where e is the electric charge of the spinning particle, Θ its Berry curvature, and E the external electric field), the motion described by Eq. (1) was dubbed the *anyonic spin Hall effect* [7,8].

Up to now, the anyonic spin Hall effect was studied for the magnetized horizons of the Kerr–Newman solution (endowed possibly with a magnetic monopole) [7,8,11], or for the Kerr black hole immersed in an aligned test uniform magnetic field [11]. However, in all these cases, the induced motion of Carrollian particles is sort of “trivial”—the particles orbit the black hole horizon on “circular trajectories” around the axis of rotation of the black hole.

The goal of the present paper is to show that a much more interesting motion of Carrollian particles is possible. To this purpose, we employ a remarkable test field in the vicinity of a rotating (Kerr) black hole, that represents an (asymptotically uniform) magnetic field whose asymptotic axis of symmetry is tilted with respect to the rotational axis of the hole [12,13]. Since a stationary black hole must be either static or axisymmetric [14], a Kerr hole immersed in a nonaxisymmetric perturbing field evolves in time—either it loses its angular momentum or its axis becomes aligned with respect to the perturbing field. For the first analysis of this effect, using scalar field but indicating also the effect for gravitational perturbing field, see [15]. The case of the external nonaligned magnetic field is quite comprehensibly analyzed in [16]. There is a number of other works, related also to the so-called Bardeen-Petterson effect, with qualitatively similar results: the component of the hole’s angular momentum orthogonal to the external field decays exponentially with the e-folding time extremely large. For the case of the magnetic field, [16] determine the e-folding time to be

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10^{10} years $(10^4 \text{Gauss}/B)^2 (10^8 M_\odot/M)$. In astrophysical situations, such a field is produced by the magnetized plasma accreting onto the black hole, with an accretion disc misaligned with the rotation axis [17]. In [16] the astrophysical applications of black hole electrodynamics are summarized in Chapter IV. For more recent references see, for example, [18–20]. As we shall see, the tilted magnetic field induces a rather curious motion of Carrollian particles. This motion is characterized by a time-dependent velocity vector field and, due to its dependence on the latitudinal coordinate, it is analogous to the “monarch butterfly migration,” e.g., [21].

The remaining part of the paper is organized as follows. In Sec. II we review the misaligned magnetic field. Section III describes the corresponding Carrollian structure induced in the horizon. In Sec. IV we analyze the motion of Carrollian particles on the horizon in the presence of the tilted magnetic field. The conclusions of our work can be found in Sec. V.

II. ROTATING BLACK HOLE IMMersed IN A MISALIGNED MAGNETIC FIELD

A rotating black hole is described by the Kerr metric, which in the standard Boyer–Lindquist coordinates reads

$$ds^2 = -\frac{\Delta}{\Sigma}(dt - a \sin^2 \theta d\phi)^2 + \frac{\sin^2 \theta}{\Sigma}(adt - (r^2 + a^2)d\phi)^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2, \quad (3)$$

where

$$\begin{aligned} A_r &= \frac{B_1 a M}{\Sigma} \sin \theta \cos \theta (r \cos \psi - a \sin \psi) + \frac{B_0 a M r}{\Sigma} (1 + \cos^2 \theta) - B_0 a, \\ A_r &= -B_1 (r - M) \cos \theta \sin \theta \sin \psi, \\ A_\theta &= -B_1 a (r \sin^2 \theta + M \cos^2 \theta) \cos \psi - B_1 (r^2 \cos^2 \theta + (a^2 - Mr) \cos 2\theta) \sin \psi, \\ A_\phi &= B_0 \sin^2 \theta \left(\frac{r^2 + a^2}{2} - \frac{a^2 M r}{\Sigma} (1 + \cos^2 \theta) \right) - \frac{B_1 \sin 2\theta}{2} \left(\Delta \cos \psi + \frac{(r^2 + a^2) M}{\Sigma} (r \cos \psi - a \sin \psi) \right). \end{aligned} \quad (7)$$

Here, B_1 denotes the field component perpendicular to the rotation axis, while B_0 is the component aligned with the axis, and ψ is the azimuthal coordinate in the Kerr ingoing coordinates:

$$d\psi = d\phi + \frac{a}{\Delta} dr. \quad (8)$$

We refer the reader to [12,13,16,26] for an analysis of the physical properties of this field and for an illustration of its dragging around the black hole.

$$\Delta = r^2 - 2Mr + a^2, \quad \Sigma = r^2 + a^2 \cos^2 \theta. \quad (4)$$

In these expressions, M stands for the mass of the hole and $J = Ma$ is its angular momentum around the axis of symmetry—the “z-axis.”

The (outer) *black hole horizon* is located at the largest root of $\Delta(r_+) = 0$, at

$$r_+ = M + \sqrt{M^2 - a^2}. \quad (5)$$

The horizon is dragged due to the rotation of the hole, and rotates with the angular velocity:

$$\Omega_+ = \frac{a}{r_+^2 + a^2}, \quad (6)$$

equal to the angular velocity of zero angular momentum observers (ZAMOs) at the outer horizon.

When $a \rightarrow M$, the black hole is called *extremal*. It possesses a degenerate horizon and its Hawking temperature vanishes. In this limit, the aligned magnetic field is expelled from the horizon, which is known as the *black hole Meissner effect* [12,13,22–24]. However, this is no longer true for the tilted magnetic field which remains to penetrate the horizon even in the extremal case [13]. (See also [25] for the case of a magnetic flux tube of a cosmic string painted on the extremal Kerr horizon, which features a penetration/expulsion phase transition depending on the ratio of the thickness of the string and the horizon radius.)

The *tilted test magnetic field* is described by the following vector potential [12,13]:

III. CARROLLIAN STRUCTURE AND MAGNETIZED HORIZON

In what follows, we will describe the Carrollian structure in the horizon of the black hole. To this purpose, we first transfer to the ingoing coordinates (v, r, θ, φ) which corotate with the horizon, by

$$\begin{aligned} d\phi &= d\varphi + \Omega_+ dv - \frac{a}{\Delta} dr, \\ dt &= dv - \frac{r^2 + a^2}{\Delta} dr. \end{aligned} \quad (9)$$

These differ from the standard Kerr ingoing coordinates (v, r, θ, ψ) by the additional $\Omega_+ dv$ term in the first expression, cf., (8), which “eliminates” the rotation of the horizon. Upon this coordinate transformation, the metric takes the following explicit form¹:

$$ds^2 = -\frac{\Delta}{\Sigma} \left(\frac{\Sigma_+}{r_+^2 + a^2} dv - \frac{\Sigma}{\Delta} dr - a \sin^2 \theta d\varphi \right)^2 + \frac{\sin^2 \theta}{\Sigma} (\Omega_+ (r_+^2 - r^2) dv - (r^2 + a^2) d\varphi)^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2, \quad (10)$$

where we have introduced

$$\Sigma_+ = r_+^2 + a^2 \cos^2 \theta, \quad (11)$$

while the new components of the vector field are now

$$\begin{aligned} A'_v &= \Omega_+ A_\phi + A_r, \\ A'_r &= A_r - \frac{a}{\Delta} A_\phi - \frac{a^2 + r^2}{\Delta} A_t, \\ A'_\theta &= A_\theta, \quad A'_\varphi = A_\phi. \end{aligned} \quad (12)$$

Note that, in these coordinates $\psi = \varphi + \Omega_+ v + \text{const.}$, where const. denotes an integration constant that will be set to zero for the remainder of our analysis, so that we have

$$\psi = \varphi + \Omega_+ v. \quad (13)$$

It is precisely this ψ which introduces the time dependence into our problem. It originates from the fact that the preferred Carrollian time v is defined in the corotating with the horizon coordinate frame.

To see this, let us define the 2-dimensional metric on the horizon and the associated Carrollian structure. The horizon is generated by a null Killing vector field, which in the new coordinates simply reads

$$\xi = \partial_v. \quad (14)$$

The corresponding v -coordinate is the (preferred) Carrollian time on the horizon.² We can also obtain a null normal to the horizon, n_μ , which is orthogonal to ∂_θ and ∂_r , such that $n_\mu \xi^\mu = 1$. It reads

¹Note that in contrast to [7], we use ingoing rather than outgoing coordinates. Consequently, in our case the Carrollian particles move on the black hole horizon rather than the white hole horizon, as is the case in [7].

²Had we chosen the standard Kerr ingoing coordinates, as opposed to the above corotating ones, we would not obtain the preferred Carrollian time, as in that case we would have $\xi = \partial_v + \Omega_+ \partial_\psi$, and the horizon metric (17) would appear as “rotating in the azimuthal direction.”

$$n = dv - \frac{a(r_+^2 + a^2)}{\Sigma_+} \sin^2 \theta d\varphi. \quad (15)$$

Using ξ and n one can define the projector into the horizon as

$$q^\mu{}_\nu = \delta^\mu{}_\nu - \xi^\mu n_\nu - n^\mu \xi_\nu. \quad (16)$$

With both indices down, this represents a degenerate metric on a $(2 + 1)$ -dimensional horizon surface:

$$q = q_{AB} dx^A dx^B = \Sigma_+ d\theta^2 + \frac{(r_+^2 + a^2)^2 \sin^2 \theta}{\Sigma_+} d\varphi^2, \quad (17)$$

with $x^A = \{\theta, \varphi\}$. The horizon \mathcal{H} can thus be endowed with a Carrollian structure [27,28]. This structure is given by a fiber bundle $p: \mathcal{H} \rightarrow S$, where S has the topology of S^2 and is just any constant v slice, $S = \mathcal{H}|_{v=v_0}$. The projection is the usual projector constructed from the metric, and the surface S is a Riemannian manifold, equipped with the metric (17). The fiber bundle has a vertical vector field given by ∂_v , which generates the vertical space—the “time” evolution along the Carrollian time v .

To obtain the magnetic field B , relevant for the motion of Carrollian particles, (1), we define [11]

$$B = \frac{1}{2} \epsilon^{AB} \hat{F}_{AB}, \quad (18)$$

where ϵ^{AB} is the Levi-Civita tensor associated with the metric q (17), $\epsilon^{\theta\varphi} = 1/\sqrt{\det q_{AB}} = 1/[(r_+^2 + a^2) \sin \theta]$, and \hat{F}_{AB} is the projection of the bulk electromagnetic field strength

$$\hat{F}_{AB} = q^\mu{}_A q^\nu{}_B F_{\mu\nu}. \quad (19)$$

In particular, for the tilted test field (7) we (upon using the corotating with horizon coordinates) obtain

$$\begin{aligned} B &= \frac{B_1 \sin \theta}{2r_+ \Sigma_+^2} (2r_+ (a^4 \cos^2 \theta - r_+^4) \cos \psi \\ &\quad + a[(r_+^2 - a^2)a^2 \cos^2 \theta + r_+^2 (3r_+^2 + a^2)] \sin \psi) \\ &\quad + \frac{B_0 (a^4 - r_+^4) \cos \theta}{\Sigma_+^2}. \end{aligned} \quad (20)$$

As a quick check let us consider various limits of this expression. First, for $B_1 = 0$, we obtain the result for the aligned magnetic field

$$B_{\text{align}} = \frac{B_0 (a^4 - r_+^4) \cos \theta}{\Sigma_+^2}, \quad (21)$$

studied in [11]. Second, in the extremal limit $a \rightarrow M$, the contribution from B_0 vanishes, and we recover

$$B_{\text{extr.}} = \frac{B_1 \sin \theta}{(1 + \cos^2 \theta)^2} (2 \sin \psi - \sin^2 \theta \cos \psi), \quad (22)$$

where $\psi = \varphi + v/(2a)$. Finally, in the Schwarzschild limit, $a \rightarrow 0$, we recover

$$B_{\text{Sch}} = -B_0 \cos \theta - B_1 \sin \theta \cos \varphi. \quad (23)$$

This is nothing but the uniform magnetic field around (spherically symmetric) Schwarzschild, written now in tilted coordinates.

IV. CARROLLIAN MOTION IN TILTED MAGNETIC FIELD

In this section we analyze the Carrollian motion induced by the tilted magnetic field, and compare our results with the motion when the magnetic field is parallel to the axis of rotation of the black hole.

Let us start by analyzing the velocity field

$$\left(\frac{d\theta}{dv}, \frac{d\varphi}{dv} \right) \quad (24)$$

induced by the magnetic field (20) via Eq. (1). This is displayed for $v = 0$ and various ratios of B_0 vs B_1 for the rotating nonextremal Kerr black hole in Fig. 1. The two limiting cases of the extremal black hole and a nonrotating black hole are for fixed ratio of B_0 and B_1 displayed in Fig. 2.³ Here, the red vectors denote the region where the norm of the velocity field is small, while yellow vectors denote large velocities. We notice that there are two vortices (centers of red whirlwinds) on opposite sides of the black hole where the velocity field vanishes and around which the Carrollian particles revolve.

In the case where only B_0 is present, the magnetic field is parallel to the axis of rotation and the Carrollian particles revolve around this axis, as previously analyzed in [11]. When B_1 is nonzero, the axis around which the Carrollian particles rotate shifts and gets positioned in an intermediary axis between the directions of B_0 and B_1 . Also notice that the motion of the particles in this case is not a perfect circle around the axis. When $B_0 = 0$, we find that the coordinates of the points at which the velocity field vanishes at $v = 0$ are located at $\theta = \pi/2$ and $\tan \phi = -a(a^2 + 3r_+^2)/(2r_+^3)$, which yields two solutions for ϕ , which differ by π . That is, the two points are antipodal, and define the axis around which the Carrollian particles orbit.

³The figures are meant for illustrative purposes only. The horizon is displayed as “spherical” and the (θ, φ) coordinates are identified with spherical angles. In particular, we ignore the fact that close to the polar regions, the horizon of a rotating black hole cannot be embedded in \mathbb{R}^3 , e.g., [29].

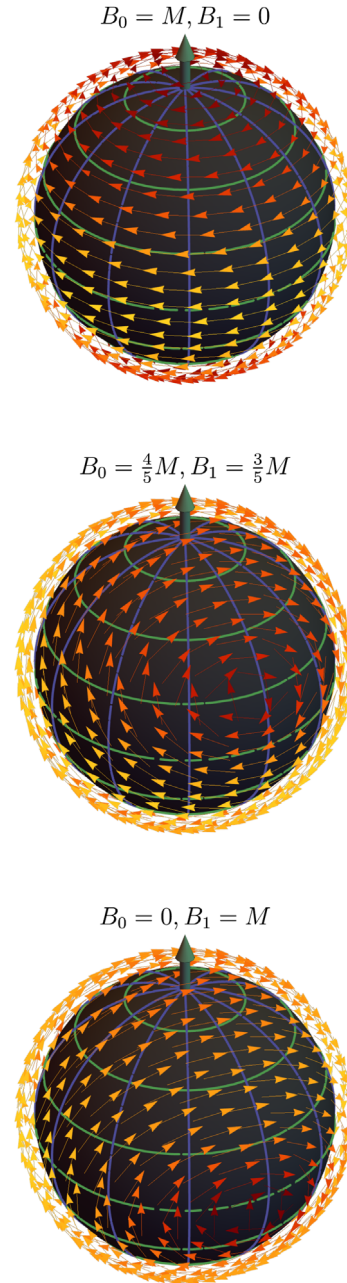


FIG. 1. *Velocity fields: field alignment.* We display velocity fields at $v = 0$ for different values of B_0 and B_1 for the nonextremal rotating Kerr black hole characterized by $r_+ = \frac{8}{5}M$ and $a = \frac{4}{5}M$. The large green arrow denotes the axes of rotation of the black hole, and the θ and φ coordinate lines are depicted in green and blue, respectively. The arrows for the vector indicate its magnitude: red stands for smaller vectors and yellow for larger ones.

Interestingly, the displayed velocity fields resemble the “eddy currents” around rotating black holes studied in the membrane paradigm in [16]. It would be interesting to probe if there is any deeper reason for this similarity.

So far we have only analyzed the velocity field at $v = 0$. As v varies, both the magnetic field and the velocity fields

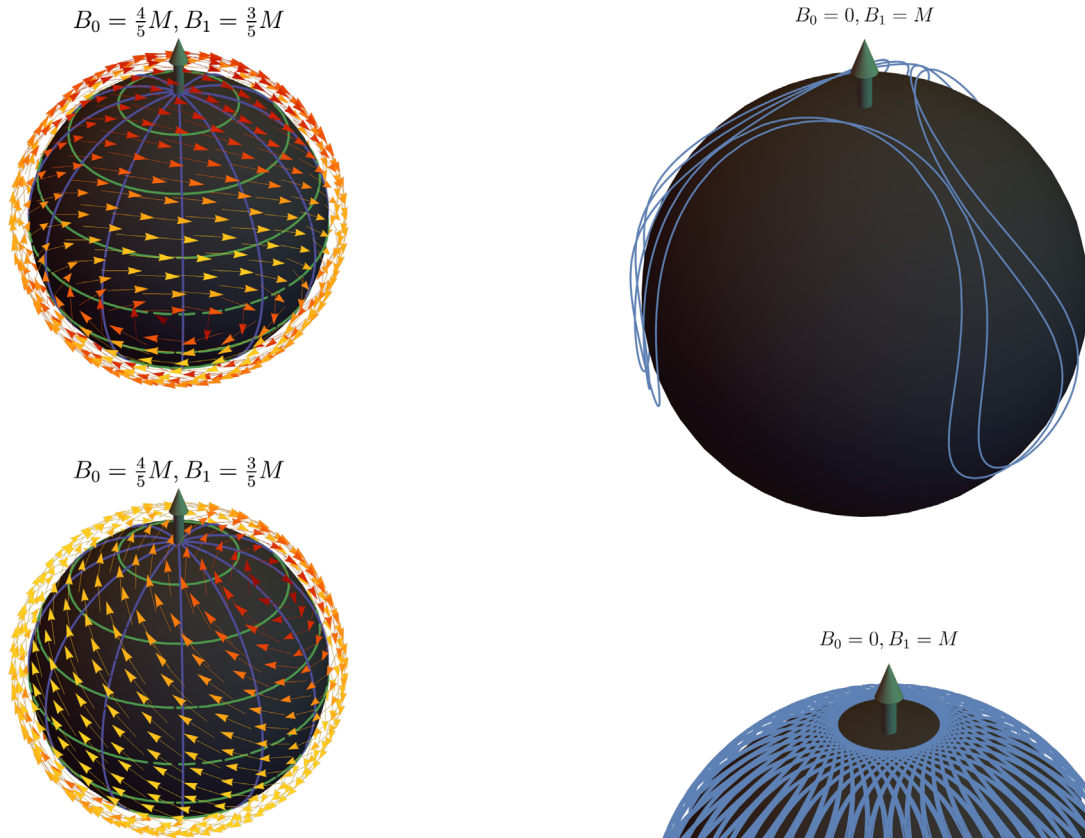


FIG. 2. *Velocity fields: effect of rotation.* We display the velocity field at $v = 0$ for a fixed ratio of $B_0 = \frac{4}{5}M$ and $B_1 = \frac{3}{5}M$ for (i) the extremal black hole (up) with $r_+ = a = M$ and (ii) nonrotating black hole with $r_+ = 2M$ and $a = 0$.

precess around the axis of rotation of the black hole with angular velocity Ω_+ , as $\varphi \rightarrow \varphi + \Omega_+ v$. This precession is caused by the dragging of the hole. Namely, while the profile of the magnetic field outside the horizon remains static with respect to observers located at infinity, the horizon itself rotates with respect to these observers with angular velocity Ω_+ , and the field seen as constant at infinity will be seen as rotating by a particle at the horizon, as is the case of the Carrollian particles discussed here.⁴

A consequence of this precession is that the motion of Carrollian particles does not simply correspond to the integral lines of the vector fields seen in Fig 1, and the nontrivial dependence on v allows for more interesting trajectories. To illustrate this explicitly, we solve the equation of motion (1) numerically for different values of initial conditions

$$\varphi(0) = \varphi_0, \quad \theta(0) = \theta_0. \quad (25)$$

⁴Let us remark, however, that while the profile of the field remains stationary outside the hole ergosphere, it is only uniform at infinity, and becomes more and more dragged along the rotation of the hole as we approach the horizon, see [26] for illustrations of this fact.

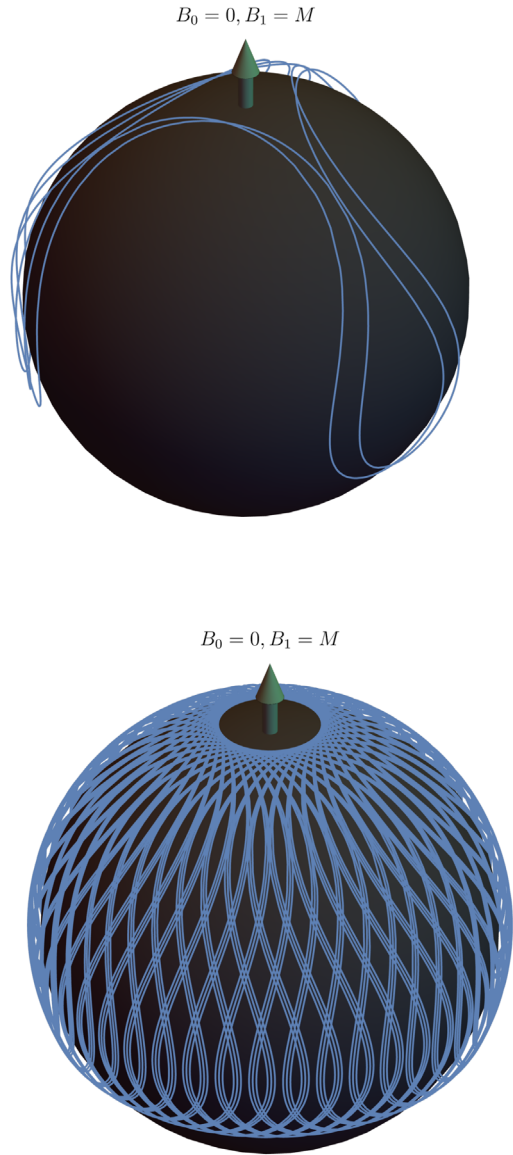


FIG. 3. *Carrollian migration on the horizon.* We display the Carrollian motion reminiscent of the monarch butterfly migration for the choice of $\theta_0 = \pi/2 = \varphi_0$ for $B_0 = M, B_1 = 0$, and $r_+ = \frac{8}{5}M, a = \frac{4}{5}M$. The two figures show the same motion of one particle; the only difference is the number of revolutions displayed.

In Fig. 3 we display the migrating motion, where Carrollian particles move from regions located around the equator to regions around the poles. The observed motion of Carrollian particles seems integrable and very similar to the motion of particles on a sphere rotating around a rotating axis, which is also not chaotic.

Finally, notice that although the curves in Fig. 3 overlap, it is not possible for Carrollian particles to collide with each other. This can be seen from the theorem of uniqueness and existence of solutions to ODEs and by noticing that the equations of motion of the Carrollian particle are of first order: A collision of two Carrollian particles would imply

that two solutions would be possible starting from the event where the collision happens, which would contradict uniqueness. As far as we can conclude, Carrollian particles on a black hole horizon in an external magnetic field follow a very organized flow that approximately rotates around a precessing axis.

V. SUMMARY

In this paper we have demonstrated that Carrollian particles can feature a rather nontrivial motion on the horizon of a black hole. To this purpose we have considered the Kerr black hole that is magnetized by a test asymptotically uniform magnetic field whose asymptotic axis of symmetry is tilted with respect to the rotational axis of the hole. As the horizon is equipped with a preferred Carrollian time, this gives rise to a “time dependent” magnetic field on the horizon and in consequence results in a (nonchaotic) “migratory” motion of Carrollian particles, which is characterized by two vortices that precess around the black hole due to the inertial frame dragging on the horizon.

It remains to be seen whether such motion of Carrollian particles (if realized in nature) would lead to some

observable features, or whether the very existence of the Carrollian particles remains only an interesting theoretical possibility.

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