

Holographic Dual of Extended Black Hole Thermodynamics

Moaathe Belhaj Ahmed¹, Wan Cong^{2,*}, David Kubizňák^{3,†}, Robert B. Mann^{1,‡} and Manus R. Visser^{4,§}

¹Department of Physics and Astronomy, University of Waterloo, Waterloo, Ontario N2L 3G1, Canada

²University of Vienna, Faculty of Physics, Boltzmanngasse 5, A 1090 Vienna, Austria

³Institute of Theoretical Physics, Faculty of Mathematics and Physics, Charles University, Prague, V Holešovičkách 2, 180 00 Prague 8, Czech Republic

⁴Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Wilberforce Road, Cambridge CB3 0WA, United Kingdom

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By respecting the conformal symmetry of the dual conformal field theory, and treating the conformal factor of the Anti-de Sitter boundary as a thermodynamic parameter, we formulate the holographic first law that is exactly dual to the first law of extended black hole thermodynamics with variable cosmological constant but fixed Newton’s constant.

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“Extended black hole thermodynamics,” also known as “black hole chemistry” [1,2], is one of the major developments in classical black hole thermodynamics in recent years. The idea stems from reconsidering the thermodynamics of asymptotically Anti-de Sitter (AdS) black holes in the context of a variable cosmological constant Λ [3,4], which is treated as a thermodynamic pressure according to the following (perfect fluid) prescription:

$$P = -\frac{\Lambda}{8\pi G_N} = \frac{(d-1)(d-2)}{16\pi G_N L^2}, \quad (1)$$

where G_N is Newton’s constant, L is the AdS radius, and d stands for the number of (bulk) spacetime dimensions.

This identification allows one to define the black hole volume V and introduces the standard pressure-volume term into black hole thermodynamics. Namely, we now have the extended first law, together with the corresponding generalized Smarr relation [5]:

$$\delta M = T\delta S + V\delta P + \Phi\delta Q + \Omega\delta J, \quad (2)$$

$$M = \frac{d-2}{d-3}(TS + \Omega J) + \Phi Q - \frac{2}{d-3}PV, \quad (3)$$

with the two being related by a dimensional scaling argument (resulting in d -dependent factors in the Smarr relation). The key result to emerge from the black hole chemistry approach is the realization that AdS black holes exhibit

phase transitions that are fully analogous to those of ordinary thermodynamic systems. In particular, one observes phase transitions à la Van der Waals [6,7], reentrant phase transitions [8], isolated critical points [9,10], superfluidlike behavior [11], and multicritical points [12]. Very recently a mechanism for the higher-dimensional origin of a dynamical cosmological constant was proposed [13].

However, the *holographic interpretation* of extended thermodynamics remained unclear for many years. The first attempts [14–18] suggested that, according to the Anti-de Sitter/Conformal Field Theory (AdS/CFT) correspondence [19], the $V\delta P$ term should be related to a $\mu\delta C$ term in the dual CFT, where C is the central charge and μ the thermodynamically conjugate chemical potential. For holographic CFTs dual to Einstein gravity the dictionary for the central charge is

$$C \propto \frac{L^{d-2}}{G_N}, \quad (4)$$

where the proportionality constant depends on the normalization of C , which is irrelevant for the discussion below. The situation is not that simple, however, as it is also standard to identify the curvature radius of the spatial geometry on which the CFT is formulated with the AdS radius L [20]. Namely, the boundary metric of the dual CFT is obtained by the conformal completion of the bulk AdS spacetime and reads [21,22]

$$ds^2 = \omega^2(-dt^2 + L^2 d\Omega_{k,d-2}^2), \quad (5)$$

where ω is an “arbitrary” dimensionless conformal factor, a function of boundary coordinates, that reflects the conformal symmetry of the boundary theory. For $k = 1$, $d\Omega_{k,d-2}^2$ is the metric on a unit $(d-2)$ -dimensional sphere. For $k = 0$,

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it is the dimensionless metric $(1/L^2) \sum_i dx_i^2$ on the plane, and for $k = -1$, it is the unit metric on hyperbolic space $du^2 + \sinh^2(u)d\Omega_{k=1,d-3}^2$.

The standard choice is to set $\omega = 1$, in which case the CFT volume \mathcal{V} is proportional to L^{d-2} . Consequently a variation of the cosmological constant in the bulk induces also a variation of the CFT volume \mathcal{V} . Hence, a pressure-volume work term, $-p\delta\mathcal{V}$, should be present on the CFT side. This implies that either (i) the corresponding CFT first law is *degenerate* as the $\mu\delta C$ and $-p\delta\mathcal{V}$ terms are not truly independent (leaving the CFT interpretation of black hole chemistry a bit obscure), or (ii) apart from varying the cosmological constant we also have to vary Newton's gravitational constant G_N so that the variations of \mathcal{V} and C are independent [20].

Alternatively, in the spirit of the second option, the authors of [23] have proposed the so-called "restricted phase space" (RPS) formalism, where the CFT volume $\mathcal{V} \propto L^{d-2}$ is kept fixed. This leaves only the $\mu\delta C$ term on the CFT side coming from a variable G_N in the bulk. The resultant holographic thermodynamics thus has nothing to do with the original black hole chemistry.

In this Letter, we generalize the approach developed in [24] to find the holographic first law that is dual to Eq. (2) while avoiding both of the above mentioned problems. Namely, in order to capture the above rescaling freedom of the CFT, while in the setting of equilibrium thermodynamics, in what follows we shall treat ω as a (dimensionless) thermodynamic parameter (similar to the horizon radius or AdS radius) rather than a function of the boundary coordinates. This will make the volume and central charge independent variables.

This is not without precedent. For the $k = 0$ planar AdS black brane case, variations of volume \mathcal{V} and central charge C are clearly independent; varying the former corresponds to changing the number of points in the system, whereas varying the latter corresponds to varying the number of degrees of freedom at each point. Since the planar case can be reached as a limit of the $k = 1$ spherical case, it is reasonable to expect this independence to extend to non-planar cases.

Consequently, rather than using the standard choice $\omega = 1$, we regard ω as another variable. This effectively amounts to changing the CFT volume, which is now proportional to

$$\mathcal{V} \propto (\omega L)^{d-2}. \quad (6)$$

In [24,25] we considered the choice $\omega = R/L$ with R being a constant boundary curvature radius, but here we allow the conformal factor to be a generic parameter that need not depend on L . For the Einstein-Maxwell Lagrangian density $\mathcal{L} = (1/16\pi G_N)(R - 2\Lambda) - \frac{1}{4}F^2$ this results in the following generalized dictionary between the bulk (without tildes) and dual CFT (with tildes) thermodynamic quantities:

$$\begin{aligned} \tilde{S} &= S = \frac{A}{4G_N}, & \tilde{E} &= \frac{M}{\omega}, & \tilde{T} &= \frac{T}{\omega}, & \tilde{\Omega} &= \frac{\Omega}{\omega}, \\ \tilde{J} &= J, & \tilde{\Phi} &= \frac{\Phi\sqrt{G_N}}{\omega L}, & \tilde{Q} &= \frac{QL}{\sqrt{G_N}}. \end{aligned} \quad (7)$$

If we now allow the bulk curvature radius L to vary, while *holding G_N fixed*, the variation of the central charge C , Eq. (4), is then purely induced by variations of L .

Analogously to the calculation in [24] (see also [25,26]), it is straightforward to show using Eq. (3) that the extended first law, Eq. (2), can be rewritten as follows:

$$\begin{aligned} \delta\left(\frac{M}{\omega}\right) &= \frac{T}{\omega}\delta\left(\frac{A}{4G_N}\right) + \frac{\Omega}{\omega}\delta J + \frac{\Phi\sqrt{G_N}}{\omega L}\delta\left(\frac{QL}{\sqrt{G_N}}\right) \\ &+ \left(\frac{M}{\omega} - \frac{TS}{\omega} - \frac{\Omega J}{\omega} - \frac{\Phi Q}{\omega}\right)\frac{\delta(L^{d-2}/G_N)}{L^{d-2}/G_N} \\ &- \frac{M}{\omega(d-2)}\frac{\delta(\omega L)^{d-2}}{(\omega L)^{d-2}}, \end{aligned} \quad (8)$$

or simply as

$$\delta\tilde{E} = \tilde{T}\delta\tilde{S} + \tilde{\Omega}\delta\tilde{J} + \tilde{\Phi}\delta\tilde{Q} + \mu\delta C - p\delta\mathcal{V}, \quad (9)$$

where, using Eqs. (4), (6), and (7),

$$\mu = \frac{1}{C}(\tilde{E} - \tilde{T}\tilde{S} - \tilde{\Omega}\tilde{J} - \tilde{\Phi}\tilde{Q}), \quad (10)$$

$$p = \frac{\tilde{E}}{(d-2)\mathcal{V}}. \quad (11)$$

The CFT first law, Eq. (9), is no longer degenerate, as both \mathcal{V} and C can now be independently varied. Together with Eqs. (10) and (11), it is exactly dual to the first law of extended black hole thermodynamics.

As is obvious from the derivation, the variation of Λ does not only enter in the variation of the central charge, but it also appears in the dictionary for the spatial volume and electric charge. The variation of Λ [the $V\delta P$ term in Eq. (2)] has thus been split into several pieces and is related to the variation of the volume, electric charge, and central charge of the CFT.

Equation (11) is the equation of state for conformal theories, which is derivable from the scaling symmetry of the CFT. Moreover, Eq. (10) is the Euler relation for holographic CFTs, which can be derived on the CFT side from the proportionality of the thermodynamic quantities with the central charge, $\tilde{E}, \tilde{S}, \tilde{J}, \tilde{Q} \propto C$, which occurs in the deconfined phase (that is, dual to an AdS black hole geometry). We note the absence of a $-p\mathcal{V}$ term in the Euler relation, which reflects the fact that the internal energy is not an extensive variable on compact spaces at finite temperature in the deconfined phase. This is not an issue, as claimed in [23], but rather a feature of holographic CFTs. In the high-temperature or large-volume regime, i.e.,

$\omega L \tilde{T} \gg 1$, the μC term becomes equal to $-p\mathcal{V}$, and hence the energy becomes extensive. As explained in [20,24], the Euler relation, Eq. (10), is dual to the Smarr formula, Eq. (3), for AdS black holes. The latter relation contains dimension dependent factors, whereas the former does not. We can understand this by expressing the PV term in the Smarr formula in terms of a partial derivative of the CFT energy:

$$-2PV = L \left(\frac{\partial M}{\partial L} \right)_{A,J,Q,G_N} = L\omega \left(\frac{\partial \tilde{E}}{\partial L} \right)_{A,J,Q,G_N}. \quad (12)$$

The boundary energy depends on the bulk quantities as $\tilde{E} = \tilde{E}[S(A, G_N), J, \tilde{Q}(Q, L, G_N), C(L, G_N), V(L, \omega)]$. Hence, the dictionary, Eqs. (4), (6), and (7), implies that

$$\begin{aligned} \left(\frac{\partial \tilde{E}}{\partial L} \right)_{A,J,Q,G_N} &= \frac{1}{L} [\tilde{\Phi} \tilde{Q} + (d-2)\mu C - (d-2)p\mathcal{V}] \\ &= \frac{1}{L} [(d-3)(\tilde{E} - \tilde{\Phi} \tilde{Q}) - (d-2)(\tilde{\Omega} J + \tilde{T} S)], \end{aligned} \quad (13)$$

where we inserted the Euler relation and the equation of state to obtain the last equality. By combining Eq. (12) and Eq. (13) and using the holographic dictionary, we recover the Smarr relation.

Since the Euler relation and equation of state follow from the scaling of the thermodynamic quantities with C and with \mathcal{V} , respectively, one can eliminate some of the terms in the first law, Eq. (9), by rescaling the CFT quantities. In particular, using Eq. (11) and rescaling some of the CFT quantities by ωL we can eliminate the $-p\delta\mathcal{V}$ term to obtain the laws

$$\delta \hat{E} = \hat{T} \delta S + \hat{\Omega} \delta J + \hat{\Phi} \delta \tilde{Q} + \hat{\mu} \delta C, \quad (14)$$

$$\hat{E} = \hat{T} S + \hat{\Omega} J + \hat{\Phi} \tilde{Q} + \hat{\mu} C \quad (15)$$

for the rescaled (dimensionless) quantities

$$\begin{aligned} \hat{E} &= \omega L \tilde{E}, & \hat{T} &= \omega L \tilde{T}, & \hat{\Omega} &= \omega L \tilde{\Omega}, \\ \hat{\Phi} &= \omega L \tilde{\Phi}, & \hat{\mu} &= \omega L \mu. \end{aligned} \quad (16)$$

The advantage of Eq. (16) is that all thermodynamic quantities are now scale invariant, and so the thermal description respects the symmetries of the CFT.

While the laws in Eqs. (14) and (15) are formally the same as those of the recently proposed RPS [23], their physical interpretation is different. In our case the CFT lives on a geometry with an arbitrary curvature radius ωL , distinct from the AdS radius L . At the same time we allow the central charge C to vary. Contrary to the RPS approach, this variation is induced by a variable cosmological constant in the bulk, rather than a variable Newton's constant.

Perhaps more interesting is to employ the Euler relation, Eq. (10), to eliminate the $\mu\delta C$ term from Eq. (9), yielding

$$\delta \bar{E} = \tilde{T} \delta \bar{S} + \tilde{\Omega} \delta \bar{J} + \tilde{\Phi} \delta \bar{Q} - \bar{p} \delta \mathcal{V}, \quad (17)$$

$$\bar{E} = (d-2) \bar{p} \mathcal{V}, \quad (18)$$

with the rescaled quantities

$$\bar{E} = \frac{\tilde{E}}{C}, \quad \bar{S} = \frac{S}{C}, \quad \bar{J} = \frac{J}{C}, \quad \bar{Q} = \frac{\tilde{Q}}{C}, \quad \bar{p} = \frac{p}{C}. \quad (19)$$

These quantities are no longer proportional to C , i.e., they are $\mathcal{O}(C^0)$. The advantage of these laws is that all thermodynamic quantities keep their correct dimensionality. Moreover, for fixed C one recovers the ‘‘standard’’ thermodynamic first law, with \bar{E} interpreted as internal energy. Of course, the rescalings in Eqs. (16) and (19) can be combined together, to obtain a dimensionless CFT law for the rescaled quantities without $-p\delta\mathcal{V}$ and $\mu\delta C$ terms. However, both the central charge C and the CFT volume \mathcal{V} can remain dynamical quantities in this first law.

To summarize, we have established an exact duality between the extended black hole thermodynamics and the CFT description. Whereas the bulk first law, Eq. (2), has one extra ($V\delta P$) term and is accompanied by a single Smarr relation, Eq. (3), to reflect the scaling symmetry of the dual CFT, the corresponding CFT first law, Eq. (9), has two extra terms and is accompanied by two relations: the Euler equation, Eq. (10), and the equation of state, Eq. (11). For this reason, the variation of Λ in the bulk corresponds to both changing the CFT central charge C and the CFT volume \mathcal{V} .

Furthermore, the corresponding $\mu\delta C$ and $-p\delta\mathcal{V}$ terms can be further eliminated by using Eqs. (10) and (11). In this way one formally recovers the laws of the RPS, or the striking Eq. (17)—but we emphasize that our approach differs from [23] in that we keep G_N fixed but allow the AdS radius L and conformal factor ω to independently vary. Finally, it would be interesting to generalize the holographic dual of extended black hole thermodynamics to other geometries, e.g., de Sitter spacetime, and other gravitational theories, e.g., higher curvature gravity.

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* wan.cong@univie.ac.at

† david.kubiznak@matfyz.cuni.cz

‡ rbmann@uwaterloo.ca

§ mv551@cam.ac.uk

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